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The Systems of Relevance Logic

Ryszard MIREK
Kraków

ABSTRACT

The system *R*, or more precisely the pure implicational fragment *R* →, is considered by the relevance logicians as the most important. The another central system of relevance logic has been the logic *E* of entailment that was supposed to capture strict relevant implication. The next system of relevance logic is *RM* or *R*-mingle. The question is whether adding m i n g l e a x i o m to *R* → yields the pure implicational fragment *RM* → of the system? As concerns the weak systems there are at least two approaches to the problem. First of all, it is possible to restrict a validity of some theorems. In another approach we can investigate even weaker logics which have no theorems and are characterized only by rules of deducibility.

1. THE SYSTEM OF NATURAL DEDUCTION

The central point of relevant logicians has been to avoid the paradoxes of material and strict implication. In other words, according to them, the heart of logic lies in the notion "if [...] then [...]". Among the material paradoxes the following are known:

M1. $\alpha \to (\beta \to \alpha)$ (positive paradox);
M2. $\sim \alpha \to (\alpha \to \beta)$;
M3. $(\alpha \to \beta) \vee (\beta \to \alpha)$;
M4. $(\alpha \to \beta) \vee (\beta \to \gamma)$.

In turn, among the strict paradoxes there are known the following:

S1. $\alpha \to (\beta \to \beta)$;
S2. $\alpha \to (\beta \vee \sim \beta)$;
S3. $(\alpha \wedge \neg \alpha) \to \beta$ (*ex falso quodlibet*).

Relevance logicians have claimed that these theses are counterintuitive. According to them, in each of them the antecedent seems irrelevant to the consequent. Generally speaking, when a formula of the form $\alpha \to \beta$ is provable in the system it should mean that "α entails β" or "β is deducible from α". In other words, it would mean semantically, as concerns material implication, that there is no assignment of values to variables which makes α true and β false, and as concerns strict implication to mean that it is impossible that α and not-β. But material implication is truth functional and it seems to be not enough to determine implication in the relevant meaning.

Let us take M1 that one can find in the Tarski–Bernays axiom system. Following Anderson and Belnap (1975), the formula violates the principle that truths entiled by necessary truths are themselves necessary. Let us admit that α is contingently true and β necessarily true. So from α we have $\beta \to \alpha$ what means that in the latter a necessity entails a contingency. In turn, if one accepts M3, one would be committed to maintain that for any two propositions one implies the other. As far as S3 is concerned, it means *ex falso quodlibet*, from a true contradiction any proposition may be deduced.

For a moment let us drop the problem of paradoxes and fallacies of implication and dwell on syntactic aspects. For this purpose we use a natural deduction system in the form proposed by Fitch (1952) but for the first time it was introduced, as a method of subordinate proofs, independly by S. Jaśkowski and G. Gentzen. Every proof within a natural deduction system begins with a hypothesis. Every subsequent step in the proof is introduced by a hypothesis or it is a formula that is derived from previous steps using one of the rules of the system. Every hypothesis introduces a subproof of the proof. Proofs and subproofs are marked out by vertical lines. Besides, the rules for each connective are rules that do not involve them. Using one of them one can introduce a formula as a hypothesis (premise). Using the other two rules it is possible to copy formulae in a proof. By this means one can copy a line of a proof within the same proof using the rule of repetition (rep) or the rule of reiteration (reit) which allows to copy a line from a proof into any of its subproofs.

Let us start from the rules for the relevance logic *R* associated with implication — the introduction rule and the elimination rule. Following Anderson and Belnap (1992), the elimination rule is:

($\to$ *E*) From $\alpha \to \beta$ and α to infer β.

The rule of implication introduction is:

($\rightarrow I$) From a proof of β on hypothesis α to infer $\alpha \rightarrow \beta$.

We are now ready to prove M1:

1.	$\vert\underline{\ \alpha\ }$	hyp;
2.	$\vert\ \ \vert\underline{\ \beta\ }$	hyp;
3.	$\vert\ \ \vert\ \alpha$	1 (reit);
4.	$\vert\ \ \beta \rightarrow \alpha$	2–3, $\rightarrow I$;
5.	$\alpha \rightarrow (\beta \rightarrow \alpha)$	1–4, $\rightarrow I$.

But it is obvious that there is no relevance between premises and they are not really used in the derivation of (4) and (5). The fallacy of relevance can be shown in the proof of S1:

1.	$\vert\underline{\ \alpha\ }$	hyp;
2.	$\vert\ \ \vert\underline{\ \beta\ }$	hyp;
3.	$\vert\ \ \vert\ \beta$	2 (rep);
4.	$\vert\ \ \beta \rightarrow \beta$	2–3, $\rightarrow I$;
5.	$\alpha \rightarrow (\beta \rightarrow \beta)$	1–4, $\rightarrow I$.

In the above proof we have $\beta \rightarrow \beta$ from the irrelevant hypothesis α. Using S1 it is possible to prove from "The Earth is round" that "Margaret is pregnant implies that Margaret is pregnant".

To resolve this problem we need additional tools that eliminate such derivations. In the classical logic some of the premises are completely irrelevant with reference to the conclusion. What we need is really using the premises in the derivation of the conclusion. For this purpose relevance logicians introduced the idea of indexing each hypothesis by a numbers. In this way each step in a proof is indexed which helps to track which conclusions depend on which hypothesis. What is more, the additional proviso for the introduction rule and some changes in the elimination rule has been introduced. They are the following:

($\rightarrow E'$) From $\alpha \rightarrow \beta_k$ and α_l to infer $\beta_{k \cup l}$
($\rightarrow I'$) From a proof of β_k on hypothesis $\alpha_{\{l\}}$ to infer $\alpha \rightarrow \beta_{k - \{l\}}$, provided l occurs in k.

Using the above new tools one may prove the *law of assertion*:

1.	$\vert\underline{\ \alpha_{\{1\}}\ }$	hyp;
2.	$\vert\ \ \vert\underline{\ \alpha \rightarrow \beta_{\{2\}}\ }$	hyp;
3.	$\vert\ \ \vert\ \alpha_{\{1\}}$	1 (reit);
4.	$\vert\ \ \vert\ \beta_{\{1, 2\}}$	2–3, $\rightarrow E'$;
5.	$\vert\ (\alpha \rightarrow \beta) \rightarrow \beta_{\{1\}}$	2–4, $\rightarrow I'$;
6.	$\alpha \rightarrow [(\alpha \rightarrow \beta) \rightarrow \beta]$	1–5, $\rightarrow I'$.

Due to the indexing system it is possible to remove the scope lines, since the numbers help to distinguish between a proof and a subproof, and it is clear in which subroof a step is contained. Then, we do not need a reiteration rule and a repetition rule. Let us consider the proof of the law of transitivity:

1. $\alpha \to \beta_{\{1\}}$	hyp;
2. $\beta \to \gamma_{\{2\}}$	hyp;
3. $\alpha_{\{3\}}$	hyp;
4. $\beta_{\{1, 3\}}$	1, 3, $\to E'$;
5. $\gamma_{\{1, 2, 3\}}$	2, 4, $\to E'$;
6. $\alpha \to \gamma_{\{1, 2\}}$	3–5, $\to I'$;
7. $(\beta \to \gamma) \to (\alpha \to \gamma)_{\{1\}}$	2–6, $\to I'$;
8. $(\alpha \to \beta) \to [(\beta \to \gamma) \to (\alpha \to \gamma)]$	1–7, $\to I'$.

The proviso included in ($\to I'$), that *l* occurs in *k*, ensures that premises are really used in the derivation of a conclusion. But what about the rules for disjunction and conjunction? As far as the truth condition for conjunction is concerned, we have the following rules:

($\wedge I$) From α_k and β_k to infer $\alpha \wedge \beta_k$;
($\wedge E$) From $\alpha \wedge \beta_k$ to infer α_k or βk.

In the natural deduction system the rules for an introduction disjunction are fairly standard:

($\vee I$) From α_k to infer $\alpha \vee \beta_k$

and

($\vee I$) From β_k to infer $\alpha \vee \beta_k$.

On the other hand, if one accepts the rule of an elimination disjunction in the following form:

($\vee E$) From $\alpha \vee \beta_k$ and $\neg \alpha_k$ to infer β_k,

it would be possible to prove unwelcomed S3:

1. $\alpha \wedge \neg \alpha_{\{1\}}$	hyp;
2. $\alpha_{\{1\}}$	1, $\wedge E$;
3. $\alpha \vee \beta_{\{1\}}$	2, $\vee I$;
4. $\neg \alpha_{\{1\}}$	1, $\wedge E$;
5. $\beta_{\{1\}}$	3, 4, $\vee E$;
6. $(\alpha \wedge \neg \alpha) \to \beta$	1–5, $\to I'$.

The solution would be to introduce some changes in ($\vee$ *E*). Anderson and Belnap appeal to a version of the disjunction elimination rule, used in G. Gentzen and D. Prawitz's natural deduction systems for intuitionistic and classical logic. In our notation the rule is:

($\vee$ *E'*) From $\alpha \vee \beta_k$, $\alpha \rightarrow \gamma_l$ and $\beta \rightarrow \gamma_l$ to infer $\gamma_{k \cup l}$.

But R. Brady (2003) formulated an alternative rule of disjunction elimination. Contrary to the Anderson and Belnap's rule, the Brady's one allows the derivation of an important principle, which is the l a w o f d i s t r i b u t i o n:

(DST) $\alpha \wedge (\beta \vee \gamma) \rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$.

The Brady's rule is of the form:

($\vee$ E^B) From $\alpha \vee \beta_k$ to infer α_k, β_k.

In the Anderson and Belnap's system (DST) is added as an additional rule but if one is endowed with the Brady's rule one is ready to prove it:

1. $\alpha \wedge (\beta \vee \gamma)_{\{1\}}$	hyp;
2. $\alpha_{\{1\}}$	1, $\wedge$ *E*;
3. $\beta \vee \gamma_{\{1\}}$	2, $\wedge$ *E*;
4. $\beta_{\{1\}}$, $\gamma_{\{1\}}$	3, $\vee$ E^B;
5. $\alpha \wedge \beta_{\{1\}}$, $\gamma_{\{1\}}$	2, 4, $\wedge$ *I*;
6. $\alpha \wedge \beta_{\{1\}}$, $\alpha \wedge \gamma_{\{1\}}$	2, 5, $\wedge$ *I*;
7. $\alpha \wedge \beta \vee \alpha \wedge \gamma_{\{1\}}$, $\alpha \wedge \gamma_{\{1\}}$	6, $\vee$ *I*;
8. $\alpha \wedge \beta \vee \alpha \wedge \gamma_{\{1\}}$, $\alpha \wedge \beta \vee \alpha \wedge \gamma_{\{1\}}$	7, $\vee$ *I*;
9. $\alpha \wedge \beta \vee \alpha \wedge \gamma_{\{1\}}$	8, ! *E*.

In the ninth step the rule (! *E*) of exclamation elimination was used which states that it is allowed to infer from $\alpha_{\{k\}}$, $\alpha_{\{k\}}$ to $\alpha_{\{k\}}$. Thus, from two lines of proof which prove the same thing we are allowed to drop one of them.

2. THE DEDUCTION THEOREM

The relationship between deductions and implications may be discussed in a semantic and a syntactic version. We are concerned with a syntactic one or more precisely with proof-theoretic version. To understand this relationship we need to formulate a metatheorem that is used to deduce proofs in a given theory. The deduction theorem states that if a formula β is deducible from α, then the implication $\alpha \rightarrow \beta$ is demonstrable or deducible from the empty set (is a theorem in a logic). Thus, for a given logic *L* we have:

If $\alpha \vdash \beta$ then $\vdash \alpha \rightarrow \beta$.

The deduction theorem may be generalized to any finite sequent:

(DT) If $\alpha_1, \alpha_2, \dots, \alpha_n, \beta \vdash \gamma$ is a valid sequent in *L*, then $\alpha_1, \alpha_2, \dots, \alpha_n \vdash \beta \rightarrow \gamma$.

Of course it is possible to infer so on until we obtain:

$\vdash \alpha_1 \rightarrow (\dots (\alpha_n \rightarrow (\beta \rightarrow \gamma)) \dots)$.

As for relevant logics what is needed in the deduction theorem is the relevance in the sequent. Thus, the sequent:

$\alpha_1, \alpha_2, \dots, \alpha_n \vdash \beta$,

is valid if and only if from the hypotheses $\alpha_{1\{1\}}, \alpha_{2\{2\}}, \dots, \alpha_{n\{n\}}$ one can derive $\beta_{\{1, \dots, n\}}$. For a valid sequent:

$\alpha_1, \alpha_2, \dots, \alpha_n, \beta \vdash \gamma$,

the inference from the assumption of $\alpha_{1\{1\}}, \alpha_{2\{2\}}, \dots, \alpha_{n\{n\}}$ and $\beta_{\{n+1\}}$ to $\gamma_{\{1, \dots, n+1\}}$ is valid as well. Using an introduction rule it is possible to infer $\beta \rightarrow \gamma_{\{1, \dots, n\}}$ from $\alpha_{1\{1\}}, \alpha_{2\{2\}}, \dots, \alpha_{n\{n\}}$, so it is obvious that the sequent $\alpha_1, \alpha_2, \dots, \alpha_n \vdash \beta \rightarrow \gamma$ is valid.

3. SYSTEMS OF RELEVANCE LOGIC

The system R, or more precisely the pure implicational fragment $R \rightarrow$, is considered by the relevance logicians as the most important and delightful. First of all, $R \rightarrow$ is the oldest one. It was formulated independently by Moh Shaw-Kwei in 1950 and A. Church in 1951. Church calls his system the "weak positive implicational propositional calculus". Following Church and Moh the axiomatic system of $R \rightarrow$ is:

$R \rightarrow 1$ $\alpha \rightarrow \alpha$ (self-identity);
$R \rightarrow 2$ $(\alpha \rightarrow \beta) \rightarrow [(\gamma \rightarrow \alpha) \rightarrow (\gamma \rightarrow \beta)]$ (prefixing)[1]

or alternatively:

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[1] Another form of p r e f i x i n g is: $(\beta \rightarrow \gamma) \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$.

$R \rightarrow 2'$ $(\alpha \rightarrow \beta) \rightarrow [(\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma)]$ (transitivity);
$R \rightarrow 3$ $[\alpha \rightarrow (\alpha \rightarrow \beta)] \rightarrow (\alpha \rightarrow \beta)$ (contraction)

or alternatively:

$R \rightarrow 3'$ $[\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$ (self-distribution);
$R \rightarrow 4$ $\alpha \rightarrow [(\alpha \rightarrow \beta) \rightarrow \beta]$ (assertion);

or alternatively:
$R \rightarrow 4'$ $[\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [\beta \rightarrow (\alpha \rightarrow \gamma)]$ (permutation).

Although the heart of relevance in *R* lies in the above implicational fragment, neither Moh nor Church considered the possibility of obtaining *R* by adding axioms for truth functions to $R \rightarrow$. Thus, the axioms for the additional truth functions have the following form:

*R*5 $(\alpha \wedge \beta) \rightarrow \alpha$;
*R*6 $(\alpha \wedge \beta) \rightarrow \beta$;
*R*7 $[(\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \gamma)] \rightarrow [\alpha \rightarrow (\beta \wedge \gamma)]$;
*R*8 $\alpha \rightarrow (\alpha \vee \beta)$;
*R*9 $\beta \rightarrow (\alpha \vee \beta)$;
*R*10 $[(\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma)] \rightarrow [(\alpha \vee \beta) \rightarrow \gamma]$;
*R*11 $[\alpha \wedge (\beta \vee \gamma)] \rightarrow [(\alpha \wedge \beta) \vee \gamma)]$;
*R*12 $(\alpha \rightarrow \neg \beta) \rightarrow (\beta \rightarrow \neg \alpha)$;
*R*13 $\neg \neg \alpha \rightarrow \alpha$.

The inference rules are:

(Adjunction) from α and β to infer $\alpha \wedge \beta$;
(Modus Ponens) given $\alpha \rightarrow \beta$, from α to infer β.

The another central system of relevance logic has been the logic *E* of entailment that was supposed to capture strict relevant implication. C. I. Lewis added a new connective to classical logic, it means the strict implication, to create the modal systems in order to avoid the paradoxes of material implication. But W. Ackermann proved that Lewis' systems do not properly interpret that α entails β and acceptance of (S3) is paradoxical as well. On the ground of Ackermans' system Π', Anderson and Belnap formulated their logic *E*. From a syntactic point of view, the system *R* is axiomatic extension of *E*.[2] The latter is a system of relevant strict implication that is both a relevance logic and a modal logic with it is necessary that α defined as follows:

$$\sim \alpha =_{def} (\alpha \rightarrow \alpha) \rightarrow \alpha.$$

[2] ANDERSON and BELNAP (1975: 340) proposed the axiomatisation where *R* is obtained from *E* by adding the axiom $\alpha \rightarrow [(\alpha \rightarrow \alpha) \rightarrow \alpha]$. This axiom is modality-destroying (demodalizer) with reference to the modality-preserving axioms.

The pure calculus of entailment $E \rightarrow$ may be axiomatised as follows:

$E \rightarrow 1$ $\alpha \rightarrow \alpha$ (identity);
$E \rightarrow 2$ $(\alpha \rightarrow \beta) \rightarrow [(\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma)]$ (transitivity)[3];
$E \rightarrow 3$ $[\alpha \rightarrow (\alpha \rightarrow \beta)] \rightarrow (\alpha \rightarrow \beta)$ (contraction)[4];
$E \rightarrow 4$ $[\alpha \rightarrow (\varphi \rightarrow \psi) \rightarrow \beta] \rightarrow [(\varphi \rightarrow \psi) \rightarrow (\alpha \rightarrow \beta)]$ (restricted permutation)[5].

In this configuration the only difference between $R \rightarrow$ and $E \rightarrow$ includes the forth axiom but the other variants of axiomatization are possible (Dunn 1986: 117–224). This is how we can prove in $R \rightarrow$ the l a w o f p e r m u t a t i o n:

1. $\alpha \rightarrow (\beta \rightarrow \gamma)_{\{1\}}$	hyp;
2. $\beta_{\{2\}}$	hyp;
3. $\alpha_{\{3\}}$	hyp;
4. $\beta \rightarrow \gamma_{\{1, 3\}}$	1, 3, $\rightarrow E'$;
5. $\gamma_{\{1, 2, 3\}}$	2, 4, $\rightarrow E'$;
6. $\alpha \rightarrow \gamma_{\{1, 2\}}$	3, 6, $\rightarrow I'$;
7. $\beta \rightarrow (\alpha \rightarrow \gamma)_{\{1\}}$	2–6, $\rightarrow I'$;
8. $[\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [\beta \rightarrow (\alpha \rightarrow \gamma)]$	1–7, $\rightarrow I'$.

But the same proof of ($R \rightarrow 4$) is impossible in $E \rightarrow$. On the other hand, a proof of the l a w o f r e s t r i c t e d p e r m u t a t i o n in $E \rightarrow$ looks as follows:

1. $\alpha \rightarrow (\varphi \rightarrow \psi) \rightarrow \beta_{\{1\}}$	hyp;
2. $\varphi \rightarrow \psi_{\{2\}}$	hyp;
3. $\alpha_{\{3\}}$	hyp;
4. $(\varphi \rightarrow \psi) \rightarrow \beta_{\{1, 3\}}$	1, 3, $\rightarrow E'$;
5. $\beta_{\{1, 2, 3\}}$	2, 4, $\rightarrow E'$;
6. $\alpha \rightarrow \beta_{\{1, 2\}}$	3–5, $\rightarrow I'$;
7. $(\varphi \rightarrow \psi) \rightarrow (\alpha \rightarrow \beta)_{\{1\}}$	2–6, $\rightarrow I'$;
8. $[\alpha \rightarrow (\varphi \rightarrow \psi) \rightarrow \beta] \rightarrow [(\varphi \rightarrow \psi) \rightarrow (\alpha \rightarrow \beta)]$	1–7, $\rightarrow I'$.

Why a proof of the l a w o f p e r m u t a t i o n is impossible in $E \rightarrow$? The reason is we could use a (minor) premise in ($\rightarrow E'$) with a smaller index than a (major) premise if a former has a form of implication. In this way a step (4) is correct but (5) is forbidden because a minor premise $\beta_{\{2\}}$ is not an implication.[6]

[3] Or alternatively prefixing.
[4] Or alternatively self-distribution.
[5] Or alternatively restricted-assertion.
[6] In terms of natural deduction with the vertical lines a formula may be reiterated if it is an implication.

Let us describe the next system of relevance logic that is called *RM* or *R*-mingle. *RM* is the axiomatic extension of *R* by the mingle axiom:

$\alpha \to (\alpha \to \alpha)$ (MA).

The question is whether adding (MA) to $R \to$ yields the pure implicational fragment $RM \to$ of the system? The answer is negative — instead of $RM \to$ one obtains $RMO \to$. The former system may be described by independent and complete set of axioms. It means that *RM* is not a conservative extension[7] of $RMO\to$. From the point of view of natural deduction adding the following *mingle rule*:

from α_k and α_l to infer $\alpha_{k \cup l}$ (MGL),

to the rules of *R*, leads to the system *RM*. But if one confines (MGL) to the form of implication:

from $(\alpha \to \beta)_k$ and $(\alpha \to \beta)_l$ to infer $(\alpha \to \beta)_{k \cup l}$ (MGL*),

and adds it to the rules of *E*, one creates the system *EM* in the form of natural deduction.

The weakest system of entailment is the logic Ticket-entailment *T*. The implicational fragment $T \to$ would be axiomatised using self-implication, prefixing, transitivy and contraction or permuted self-distribution.

In the system of natural deduction it is necessary to modify ($\to E'$) in the following form:

($\to E^*$) From $\alpha \to \beta_k$ and α_l to infer $\beta_{k \cup l}$, provided $\max(k) \leq \max(l)$.

Thus, a proof of prefixing would look as follows:

1. $\alpha \to \beta_{\{1\}}$	hyp;
2. $\gamma \to \alpha_{\{2\}}$	hyp;
3. $\gamma_{\{3\}}$	hyp;
4. $\alpha_{\{2, 3\}}$	2, 3, $\to E^*$;
5. $\beta_{\{1, 2, 3\}}$	1, 4, $\to E^*$;
6. $\gamma \to \alpha_{\{2\}}$	3–4, $\to I'$;
7. $\gamma \to \beta_{\{1, 2\}}$	3–5, $\to I'$;
8. $(\gamma \to \alpha) \to (\gamma \to \beta)_{\{1\}}$	6–7, $\to I'$;
9. $(\alpha \to \beta) \to [(\gamma \to \alpha) \to (\gamma \to \beta)]$	1–8, $\to I'$.

[7] A logical theory T_2 is a (proof theoretic) conservative extension of a theory T_1 if the language of T_2 extends the language of T_1 and every theorem of T_1 is a theorem of T_2 and any theorem of T_2 which is in the language of T_1 is already a theorem of T_1.

4. SEMANTICS FOR RELEVANCE LOGICS

There are known four basic semantics which are used for relevance logics. The best known is the Routley–Meyer semantics that is called the relational semantics. The remaining semantics do not use relations. There is A. Urquhart's semilattice semantics, K. Fine's operational semantics and J. M. Dunn's algebraic semantics. We fix one's attention on the relational semantics.

The starting point for *R*. Routley and B. Meyer was the possible world semantics for modal logic. But it was needed to modify the semantics to fit relevant logic. For this purpose, they introduced a three place (ternary) accessibility relation instead of the binary accessibility relation. Thus, in modal logics we have possible worlds but with regard to relevant logics we have situations.[8] In contradiction to worlds, situation can be incomplete and inconsistent. Incompletness means that in some situations we does not have information whether a given proposition is true or false. By a positive relational frame for R_+ without negation we mean a triple ⟨*K*, *R*, *O*⟩, where *K* is non-empty set of situations (Routley called the elements *a*, *b*, *c* etc. in *K* set-ups), *R* is three-placed relation on *K* and $O \in K$ is the set of logical situations.[9] We may interpret *Rabc* as the combination of the pieces of information *a* and *b* which are a piece of information in *c*. For the framework we have the following requirements:

1. *ROaa* (identity);
2. $Rabc \Rightarrow Rbac$ (commutativity);
3. $R^2(ab)\ cd \Rightarrow R^2a(bc)d$ (associativity);
4. *Raaa* (idempotence);
5. *Rabc* and $a' \leq a \Rightarrow Ra'bc$ (monotony).

By ≤ we mean the hereditariness relation. This a binary relation on situation and $a \leq b$ means that a situation *b* extends a situation *a*. The relation is reflexive, transitive and anti-symmetrical, so it is a partial order that is defined as follows:

$a \leq b$ if and only if there is some logical situation that *ROab*.

For *R* we admit the following notation as it concerns an arity of a relation:

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[8] For more philosophical interpretation see MARES 2007: 34.

[9] *Three-Termed Relational (Routley–Meyer) Semantics for* R_+, in: ANDERSON, BELNAP 1992.

R^0ab if and only if $a \leq b$;
R^1abc if and only if $Rabc$;
R^2abcd if and only if $\exists x(Rabx \,\&\, Rcdx)$;
R^3abcde if and only if $\exists x(R^2abcx \,\&\, Rxde)$ etc.

In the system of natural deduction the subscripts refer to situations. If we have the structure $\alpha_{\{1\}}$, we interpret it that there is some situation, let us say a^1, in which α is true. For the structure $\alpha_{\{1,\,2\}}$ the subscript refers to some arbitrary situation b such that Ra_1a_2b, and for $\alpha_{\{1,\,2,\,3\}}$ there is an arbitrary situation b such that $R^2a_1a_2a_3b$ etc. The ternary relation R seems to be a little complicated. According to recent work by Priest, Sylvan and Restall (2002: 1–129), this interpetation is reminiscent of that of non-normal modal logics. There are two sorts of situations in a frame — normal ones and non-normal ones. But, contrary to modal operators, the truth conditions for connectives in relevance logic are the same through the frame. Normal points are given in the interpretation of implication in modal logic S5:

$a \models \alpha \to \beta$ if and only if for every b, if $b \models \alpha$, then $b \models \beta$.

On the other hand, to interpret the ternary relation R for implication, we have non-normal points:

$a \models \alpha \to \beta$ if and only if for every b and c where $Rabc$ if $b \models \alpha$, then $c \models \beta$ ($\to$).

A positive relational model is quadruple $\langle K, R, O, \models\rangle$, where $\langle K, R, O\rangle$ is a positive relational frame and $\models$ is a relation from K to sentences of R_+ satisfying the following atomic hereditary condition:

For a propositional variable p, if $a \models p$ and $a \leq b$, then $b \models p$.

For any formulae α and β we have the following valuation clauses, including ($\to$) as well:

$a \models \alpha \wedge \beta$ if and only if $a \models \alpha$ and $a \models \beta$ ($\wedge$);
$a \models \alpha \vee \beta$ if and only if $a \models \alpha$ or $a \models \beta$ ($\vee$).

Using the accessibility relation R Routley and Meyer established the semantic counterparts of axioms:

1. $R0aa$;
2. if $R^2 0abc$, then $Rabc$;
3. if $R0ab$ and $R0bc$, then $R0ac$;
4. if R^2abca, then there exists such $x \in K$, that $Rbcx$ and $Raxa$;
5. if $Rabc$, then R^2abbc;
6. if R^2abca, then there exists such $x \in K$, that $Racx$ and $Rbxa$;

7. $Ra0a$;
8. if $Rabc$, then $Rbac$.

In this way one can distinguish the positive models adding the above conditions:

T_+ : the accessibility relation R fulfils the conditions 1–6;
E_+ : the accessibility relation R fulfils the conditions 1–7;
R_+ : the accessibility relation R fulfils the conditions 1–8.

If one adds to R_+ the following condition:

9. $R00a$,

then obtains RM_+.

5. WEAK SYSTEMS OF RELEVANCE LOGIC

There are at least two approaches to the problem of weak systems. First of all, it is possible to restrict a validity of some theorems. Thus, in the logic *S* with a single binary connective → of Martin and Meyer (1982: 869–887) we have just two axioms, that is:

(1) $(\beta \to \gamma) \to (\alpha \to \beta) \to (\alpha \to \gamma)$ prefixing,
(2) $(\alpha \to \beta) \to [(\beta \to \gamma) \to (\alpha \to \gamma)]$ suffixing,

and the rule of Modus Ponens. The system has no theorems of the form $\alpha \to \alpha$, which means that all inferences from itself are invalid. In this way the system *S* rejects the traditional fallacy of circular reasoning.

Another interesting system is the Brady's content semantics (Brady 2003) or the system Dj^d. According to E. Mares, a language of this system is not a formal one, but rather an "interpretational language", that is a language that is already interpreted. If we admit that *x* is a set of sentences then $c(x)$ is an analytic closure of the set *x* of sentences or the content of the sentences. For instance, the content $c(x)$ of the sentence *c* "John is a bachelor" is the sentence "John is unmarried".

The relation of content containment, $\supseteq$, is the superset relation, and if *x* and *y* are the sets, then $x \supseteq y$ if and only if $y \subseteq x$. In a formal language, besides an implication, we have conjunction, disjunction and of course propositional variables and parentheses. The content of a disjunction $c(x \text{ or } y)$ is the intersection of the contents $c(x) \cap c(y)$ and the content of a conjunction $c(x \text{ and } y)$ is the content of the unions of the contents of each conjunct, that is $c(c(x) \cup c(y))$. The above interpretation seems, at first sight, wrong, but it is proper from the point of view of the content semantics. Following E. Mares, let us take as an example the disjunction "Table *x* is made of rimu or table *x* is 12 feet wide". In the content of "Table *x* is made of rimu" is the sentence "This table is made of wood", but not "Table *x* is more than 11 feet wide". And vice versa, in the content "Table *x* is 12 feet wide" is the sentence "Table *x* is more than 11 feet wide", but not the sentence "Table *x* is made of wood". Thus, neither belongs in the content of the disjunction, but their disjunction belongs in the content.

In Brady's logic we get the following weaker version of transitivity:

(3) $[(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \gamma)] \rightarrow (\alpha \rightarrow \gamma)$,

and the law of contraction is not valid:

(4) $[(\alpha \rightarrow (\alpha \rightarrow \beta)] \rightarrow (\alpha \rightarrow \beta)$.

(4) is deductively equivalent to the Modus Ponens theorem:

(5) $\alpha \wedge (\alpha \rightarrow \beta) \rightarrow \beta$.

The antecedent of (5) is a union of the contents of a sentence and a containment, that is $c(\alpha) \cup c((c)\alpha \rightarrow c(\beta))$, and the conclusion is a content of sentence $c(\beta)$. (5) is the theorem (axiom) but not rule form of Modus Ponens, and there is no guarantee that the $c(\beta)$ is contained in $c(\alpha)$, because $(c)\alpha \rightarrow c(\beta)$ would be not true. What is more, $c(\alpha)$ is not necessarily contained in the content of the containment sentence, $(c)\alpha \rightarrow c(\beta)$. So, $c(\beta)$ is not generally contained in $c(\alpha) \cup (c)\alpha \rightarrow c(\beta)$.

However, this is not the only possible approach. We can investigate even weaker logics which have no theorems and are characterized only by rules of deducibility. In this way D. M. Gabbay (1976) introduced systems $\vDash_0$ and $\vDash_1$. But first, following Wójcicki (1984), let us describe how to define a d e d u c t i v e s y s t e m from the set of theorems of some logical system. It is possible to represent the notion of entailment $\rightarrow$ by the notion of deduction, that is by the metalogical connective $\vdash$. Then, we can have the following definition of a deductive system:

The formulas $\alpha_1, \alpha_2, \ldots, \alpha_n \in Fm$ entails $\alpha \in Fm$ if and only if the formula $\alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_n \rightarrow \alpha$ is a t h e o r e m of *R*, *RM*, *E*.

This determines the consequences of only nonempty and finite sets of formulas. Wójcicki adds the conditions that the logic must be f i n i t a r y and have no theorems. The systems $\vDash_0$ and $\vDash_1$ have no theorems and are equal. In the extension of $\vDash_1$, that is $\vDash_2$, Bradley tries to find out what conditions would give $\rightarrow$ the meaning of strict implication or intuitionistic implication.

The last system is $\models_3$ that corresponds to B^+. In this last system we have the weaker form of prefixing and suffixing:

$(\beta \to \gamma) \vdash (\alpha \to \beta) \to (\alpha \to \gamma)$ prefixing (rule),
$(\alpha \to \beta) \vdash [(\beta \to \gamma) \to (\alpha \to \gamma)]$ suffixing (rule).

The interesting algebraic study of a deductive system we can find in the paper of Font and Rodriguez (1994). They define a deductive system *WR* which corresponds to the semantic entailment associated with the relational models of Routley, Meyer, Fine and others. *R* is an axiomatic extension of *WR*. Following the idea, it is possible to define the other deductive systems like *WRM*, *WE* etc., and the systems *RM* and *E* etc. are axiomatic extensions of them. Generally we can say:

Definition 1: We call $WR = \langle Fm, \vdash_{WR}\rangle$, $WRM = \langle Fm, \vdash_{WRM}\rangle$, $WE = \langle Fm, \vdash_{WE}\rangle$, the deductive systems defined by the condition that, for any $\Gamma \cup \{\alpha\} \subseteq Fm$, $\Gamma \vdash_{WR} \alpha$, $\Gamma \vdash_{WRM} \alpha$, $\Gamma \vdash_{WE} \alpha$, if and only if there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Gamma$ such that consequently $\vdash_R \alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_n \to \alpha$, $\vdash_{RM} \alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_n \to \alpha$, $\vdash_E \alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_n \to \alpha$. Alternatively we can say the following:

1. $\vdash_{WR}$, $\vdash_{WRM}$ and $\vdash_{WE}$ are finitary.
2. *WR*, *WRM* and *WE* have no theorems.
3. For every $\alpha_1, \alpha_2, \ldots, \alpha_n \in Fm$, consequently $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \vdash_{WR} \alpha$ if and only if $\vdash_R \alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_n \to \alpha$, $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \vdash_{WRM} \alpha$ if and only if $\vdash_{RM} \alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_n \to \alpha$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \vdash_{WE} \alpha$ if and only if $\vdash_E \alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_n \to \alpha$.

We can prove that the following rules:

(R1) $\alpha, \alpha \to \beta \vdash \beta$ Modus Ponens,
(R2) $\alpha, \beta \vdash \alpha \wedge \beta$ Adjunction,

are also the rules of *WRM*, *WE* and *WR*. (R1) follows from the Slaney's theorem:

$[\alpha \wedge (\alpha \to \beta)] \to \beta$.

(R2) follows from the axioms of *R*, *RM* and *E*:

(Axiom 1) $\alpha \to \alpha$,
(Axiom 2) $\alpha \wedge \beta \to \alpha$,
(Axiom 3) $\alpha \wedge \beta \to \beta$.

It is known that *R* and *RM* are both algebraisable, but *E* is not. As far as *WRM*, *WE* and *WR* are concerned, neither of them are algebraisable. It is obvious if we consider the algebra 2 for any deductive system without theorems. There are three filters on it, namely ∅ (empty set), because a deductive system has no theorems, {1} — by the completeness theorem, and trivially D e f i n i t i o n 2: By $\Omega_A F$ (the Leibniz's operator) we mean the largest congruence of any algebra A compatible with a filter *F*. Thus, the whole algebra is the congruence compatible with ∅, i.e. $\Omega_2\varnothing = 2 \times 2$. Then, the identity relation I_2 is the congruence compatible with {1}, i.e. $\Omega_{2\{1\}} = I_2$, and $\Omega_2 2 = 2 \times 2$. Thus, we have $\varnothing \subseteq \{1\}$ while $\Omega_2\varnothing \not\subset \Omega_{2\{1\}}$. So such a system is not protoalgebraic and because any algebraisable deductive system is protoalgebraic, therefore *WRM*, *WE* and *WR* are not algebraisable.

6. SUMMARY

It is supposed that system *R* the most properly captures relevant implication, and *E* is supposed to capture as well the notion of strict relevant implication. It is possible to add a necessity operator to *R*. According to Ackermann, to say that "α entails β" means that "logical connection holds between α and β". On the other hand, Parry's system of a n a l y t i c i m p l i c a t i o n (*analytische Implikation*) develops Kant fs dictum that the 'predicate is contained in the subject'. In this system $\alpha \to \beta$ is provable if all variables in β also occur in α. In consequence the Lewis paradoxes S2 and S3 fail. But it is possible to distinguish between prevalid and valid sequents/sets of sentences. The inference from α to β is prevalid if $\alpha \models \beta$ (β is a semantic consequence of α), and for no proper subsets of α and β we have $\alpha' \models \beta'$. The Lewis paradoxes fail because the proper subsets respectively there are $\alpha \wedge \neg\, \alpha$ / ∅ and ∅ / $\beta \vee \neg\, \beta$. In Parry's system collapses the notion of validity and prevalidity contrary to such systems as *R* and *E*, so these systems are more elegant.

There is close connection between relevant entailment and conditionals because they express a connection of relevance between the antecedent and consequent of true conditional. A false antecedent or true consequent are insufficient to guarantee the truth of a conditional. It seems that the Routley–Meyer ternary relation would be proper to give a truth condition for the conditional. When we evaluate the conditional we need to consider circumstances in which the antecedent and the consequent are true. For the antecedent one considers the set of circumstances in which it is true. But what about the consequent? It is possible to consider the set of circumstances in which it fails. Out of discussion is the point that the antecedent helps to evaluate the conditional, but the problem whether the consequent helps is still open.

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